Spring 2007
COMPUTER SCIENCES DEPARTMENT
UNIVERSITY OF WISCONSIN-MADISON
PH. D. QUALIFYING EXAMINATION

Optimization

Monday, February 5, 2007
3:00-7:00 PM

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.

2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On one of your books list the numbers of all the questions answered. *Do not write your name on any answer book.*

3. Return all answer books in the folder provided. Additional answer books are available if needed.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

OPTIMIZATION

Spring 2007 Qualifying Exam
February 5, 2007
Instructions: Answer any 5 of the following 8 questions.

1. Consider the following linear program:

$$
\begin{array}{rl}
\min & 3x_1 + 2x_2 - x_3 \\
\text{subject to} & 2x_1 + x_2 - x_3 \geq 5 \\
& 6x_1 + 7x_2 - 5x_3 \geq 10 \\
& -x_1 - x_2 + x_3 \geq -2 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

   (a) Write down the dual linear program.

   (b) Show that $x^* = (3,0,1)$ is an optimal solution. Do not use the simplex method.

   (c) Let b_2 refer to the right hand side of the second primal constraint (note that in the above formulation, this value is 10). Assuming that no other data change, what is the largest value of b_2 for which x^* is optimal? Justify your response.

   (d) Assuming that no other data change, what is the largest integer value of b_2 for which x^* is the *unique* optimal solution? Justify your response.

   (e) Assuming that no other data change, what is the largest integer value of b_2 for which u^*, the *unique dual* optimal solution of (a), remains the unique optimal solution? Justify your response.

2. Let $a_0, \ldots, a_m$ be points of $\mathbb{R}^n$. Suppose that for each $x^* \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$, if

$$\langle x^*, a_i \rangle \leq \xi, \qquad i = 1, \ldots, m$$

   then also $\langle x^*, a_0 \rangle \leq \xi$. Prove that a_0 must belong to the convex hull of $a_1, \ldots, a_m$.

3. For a newsboy who sells papers on a street corner, the demand is uncertain, and the newsboy must decide how many papers to buy from his supplier. If he buys too many papers he is left with unsold papers that only have a small salvage value at the end of the day; if he buys too few papers he has lost the opportunity of making a higher profit.

   From past history, the newsboy makes probability estimates $p(s)$ for each of a finite number of demand scenarios $d(s)$.

   Suppose that the newsboy gets discounting when he purchases newspapers from his supplier so that his purchase price for x newspapers is given below:

$$\begin{array}{ll} 25x & \text{if } 0 < x \leq 10 \\ 20x & \text{if } 10 < x \leq 20 \\ 15x & \text{if } 20 < x \end{array}$$

   Suppose further that his salvage value is 10, and he can sell newspapers for 30. Write down a GAMS or AMPL model to determine how many newspapers he should purchase to maximize his expected profit.

4. Consider the following radiation treatment planning problem: We wish to construct a radiation delivery aperture comprised of a number of row segments chosen from rows $i = 1,2,\ldots,m$. For a given row i, row segment j is denoted by s_{ij} and is comprised of a set of positions in row i, and has an associated quality index $q_{ij} > 0$. A null row segment denoted by n_i with quality index 0 is also defined for each row i. Delivery constraints are defined on adjacent rows such that if segment s_{ij} may be used in conjunction with segment $s_{i+1,k}$, then that pair is said to be allowable. A pair of rows that is not allowable may not both be used in an aperture. Delivery constraints also dictate that the set of selected rows containing non-null row segments must be a single set of consecutive rows. The goal is to determine a set of consecutive rows and corresponding row segments satisfying the delivery constraints so as to achieve the largest total quality $\sum q_{ij}$.

   (a) Model this problem as a shortest path problem and sketch the corresponding network. (Clearly state the correspondence between the node-arc sets used in the model and the original problem. Hint: the delivery constraints dictate that aperture construction must terminate once a null row segment is added after a non-null row segment.)

   (b) Is the resulting network acyclic, and what is the significance of acyclicity in this context?

(c) Discuss how an optimal aperture can be constructed from an appropriate solution of the network optimization problem. (What assumptions are you making about the nature of the optimal solution and what is the justification for these assumptions?)

5. Consider the following set:

$$Z = \left\{ (s,y) \in \mathbb{R} \times \mathbb{Z}^2 : \begin{array}{r} s + 10y_1 \geq 5 \\ s + 10y_2 \geq 2 \\ s, y_1, y_2 \geq 0 \end{array} \right\}$$

   (a) Prove that each of the inequalities $s + 5y_1 \geq 5$ and $s + 2y_2 \geq 2$ are valid for Z.

   (b) Do the valid inequalities above define facets or the convex hull of Z? Justify your answer.

   (c) Consider the following inequality:

$$s + 3y_1 + 2y_2 \geq 5$$

   Is this a valid inequality for Z? Why or why not? (In answering, it may be helpful to remember that $s \geq 0$ and that $s + 2y_2 - 2$ is always nonnegative.)

   (d) Does the inequality above define a facet? Why or why not?

6. In knowledge-based machine learning and data mining one needs to incorporate implications such as:

$$g(x) \leq 0,\ x \in R^n \implies \theta(x) \geq 0, \qquad (*)$$

as the following constraint:

$$\exists v \in R^m,\ v \geq 0 :\ v'g(x) + \theta(x) \geq 0,\ \forall x \in R^n. \qquad (**)$$

Here, $g : R^n \to R^m$, $\theta : R^n \to R^1$ and the prime denotes the transpose.

   (a) Under what assumptions does $(**) \implies (*)$?

   (b) Under what assumptions does $(*) \implies (**)$?

7. Suppose that $C$is a convex subset of $\mathbb{R}^n$.

- Define a face of C.
- Assume that for some $x_0^* \in \mathbb{R}^n$ the infimum on C of the linear functional $\langle x_0^*, \cdot \rangle$ is actually attained at a point $x_0 \in C$. Show that the set $X := \{x \in C \mid \langle x_0^*, x \rangle = \langle x_0^*, x_0 \rangle\}$ is a face of C.
- If D is a convex subset of C and a point of the relative interior of D lies in X, what can you say about the range of values of the functional $\langle x_0^*, \cdot \rangle$ on D? Justify your assertion(s).

8. Consider the equality constrained optimization problem

$$\min f(x) \text{ subject to } c(x) = 0,$$

where $f : R^n \to R$ and $c : R^n \to R^m$ are twice continuously differentiable functions. Let x^* be a point satisfying Karush-Kuhn-Tucker conditions with Lagrange multipliers λ^*, and suppose that second-order sufficient conditions are satisfied at (x^*, λ^*), so that x^* is a strict local solution for the problem above. Consider also the following augmented Lagrangian:

$$\hat{\mathcal{L}}(x, \lambda; \mu) = f(x) - \lambda^T c(x) + \frac{\mu}{2} \|c(x)\|_2^2,$$

where μ is a positive parameter.

Show that there is a threshold value $\bar{\mu}$ such that for all $\mu > \bar{\mu}$, x^* is a strict local minimizer of the function $\hat{\mathcal{L}}(x, \lambda^*; \mu)$.