Theory Qual

Spring 2008

Please answer all 4 questions below.

1. Show that the following problem is NP-complete: Given an undirected graph G with colored edges, two distinguished vertices s and t, and an integer k, decide whether there exists an st-cut that consists of edges of no more than k distinct colors.

2. Devise a logarithmic-space algorithm for constructing a minimum spanning tree for a given weighted undirected graph. You can make use of the logarithmic-space algorithm for undirected st-connectivity.

3. You are given n balloons, each having a certain capacity for holding air. For every $i \in \{1, 2, \ldots, n\}$, there is exactly one balloon with a capacity of $1/i$ units. The balloons look identical, and there is no way to tell how much capacity any one has by just looking at it. Your goal is to inflate the balloons to the maximum total volume possible. At every step, you can pick a balloon and decide how much air to fill in it. If this quantity does not exceed the capacity of the balloon, you get a reward equal to the amount of air you filled. If it exceeds the capacity, the balloon pops and you get nothing, but you find out the true capacity of that balloon. Each balloon can be inflated at most once.

   (a) Exhibit a deterministic strategy that guarantees a reward of 1. One can actually show that no deterministic strategy can guarantee a reward strictly larger than 1, but we won't ask you to prove that today.

   (b) Find a randomized strategy that guarantees an expected reward of at least $\sum_{i=1}^{n} 1/i^2$. You'll get partial credit for a randomized strategy that guarantees an expected reward strictly larger than 1 for $n > 1$.

4. Logarithmic advice can often be efficiently eliminated. More precisely, for several complexity classes $\mathcal{C}$ the following statement holds:

$$\mathcal{C} \subseteq \mathrm{P}/O(\log n) \quad \Rightarrow \quad \mathcal{C} = \mathrm{P}.$$

   Prove the statement for (a) $\mathcal{C} = \mathrm{NP}$, (b) $\mathcal{C} = \mathrm{AM}$, and (c) $\mathcal{C} = \mathrm{EXP}$.

G O O D   L U C K ! !