Theory Qualifying Exam
Spring 2009

Directions. You have four hours. There are 4 problems, please do them all. If you cannot completely solve a problem, we will award partial credit for work that is correct and relevant to the question.

1. This problem deals with strings over the 2 symbol alphabet $\{H, T\}$.
   a) Give a deterministic finite automaton that recognizes the set of strings ending in HHH. Do the same for the strings ending in HTH.
   b) Suppose that the symbols are generated by flipping a fair coin. Determine the expected number of coin flips to generate the sequence HHH. What about the sequence HTH?

2. Roads in the city of Winterville are organized as a tree. Every snowy morning a snow plow goes around clearing roads in the city with the goal of clearing as many intersections as possible. It starts from a depot at the root of the tree and returns to the depot at the end of the day.
   a) When the roads are really bad, the snow plow cannot clear all intersections during the day. Give a strongly polynomial time algorithm to determine what route it should take to clear as many as possible by a certain deadline. Assume that you are given the precise (integral) amount of time it will take the plow to go over each road.
   b) The city decides to employ 2 snow plows and wants to determine the minimum time it will take to clear all intersections (measured by the time the last plow takes to return to the depot). Prove that this problem is NP-hard.

3. Let $H = (h_{i,j})$ be a $k \times k$ integer symmetric matrix. Let $G = (V, E)$ denote an undirected graph. The *Partition Function* is defined by the formula

$$Z(H, G) = \sum_{\sigma: V(G) \to \{1, \ldots, k\}} \prod_{(u,v) \in E(G)} h_{\sigma(u), \sigma(v)}.$$

   The computational problem is to compute $Z(H, G)$ given the inputs H and G.
   a) Can you express the problem of counting the number of valid (vertex) colorings of a graph (using 3 distinct colors) in this framework?
   b) Show that the problem is #P-hard even restricted to the case $k = 2$.
   c) Prove that if H has rank at most one, then $Z(H, G)$ can be computed in polynomial time.

4. This deals with the job assignment problem. Recall that the input for an assignment problem is an $n \times n$ matrix of positive numbers $c_{i,j}$, and the task is to find a permutation σ of $\{1, \ldots, n\}$ that maximizes

$$\sum_i c_{i,\sigma(i)}.$$

Intuitively, $c_{i,j}$ is the benefit gained by assigning worker i to job j, and we seek a matching of workers to jobs that maximizes total benefit.

a) The matrix has *Property M* if, for all $1 \leq i, j < n$, the following inequality holds:

$$c_{i,j} + c_{i+1,j+1} \geq c_{i,j+1} + c_{i+1,j}.$$

Give a simple "greedy" algorithm to find an optimal assignment for matrices having property M. Your algorithm should have running time $O(n)$, assuming that array elements can be accessed in unit time. (Hint: Property M implies more inequalities.)

b) Suppose the matrix entries are i.i.d. from the uniform distribution on the interval $[0, 1]$. Let P_n be probability that the matrix has property M. Show that

$$\lim_{n \to \infty} P_n = 0.$$