Theory Qual

Spring 2013

Please answer all questions below.

1. You are given n nuts and n bolts, all of different sizes. Each bolt matches exactly one nut. Your goal is to find the biggest bolt.

   Consider the following randomized algorithm. Fix a random permutation of the bolts and a random permutation of the nuts. At every step you compare a bolt and a nut, starting with the first ones from each permutation. If the bolt is smaller, you discard it and move on to the next one in the bolt permutation. Otherwise, you discard the nut and move on to the next one in the nut permutation. The process ends when all nuts are discarded, at which point the current bolt is the biggest one.

   Compute exactly the expected number of comparisons this process makes.

2. Show that the language of all pairs of isomorphic graphs is in coNP/poly.

3. For a given flow network $G = (V, E)$, we can classify the vertices into three types:

   - upstream vertices belong to the source side S of every minimum-capacity cut (S, T),
   - downstream vertices belong to the sink side T of every minimum-capacity cut (S, T), and
   - mid-stream vertices belong to the source side S of some minimum-capacity cut (S_1, T_1) and to the sink side T_2 of some other minimum-capacity cut (S_2, T_2).

   Give an efficient algorithm that takes a network and returns the set of all upstream, mid-stream, and downstream vertices.

4. Let M be a deterministic finite automaton over the alphabet Σ with transition function δ_M. Assume that all states are reachable from the start state s_0. Construct the nondeterministic automaton N as follows: Reverse all transitions in M, i.e., $p \in \delta_N(q, a)$ iff $\delta_M(p, a) = q$, for states p, q of M and $a \in \Sigma$. We make s_0 the unique accepting state of N. (We will not need to specify the start state of N.) Apply the standard power set transformation to turn N into a deterministic finite automaton N': For any subset P of states of N and any $a \in \Sigma$, define

   $$\delta_{N'}(P, a) = \bigcup_{p \in P} \delta_N(p, a).$$

   Choose the set of all accepting states of M as the start state S_0 of N', and remove all states of N' that are not reachable from S_0. Denote the resulting deterministic finite automaton by D.

   (a) What is the relationship between $L(M)$ and $L(D)$?

   (b) Show that D is minimal, i.e., there does not exist a deterministic finite automaton that accepts the same language as D and has fewer states than D.

G O O D L U C K ! !