Spring 2013 Qualifier Exam: OPTIMIZATION

4 February 2013

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 out of 6 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Suppose D^1, D^2, P^1 and P^2 are finite sets of (directions and) points in $\mathbf{R}^n$, with $0 \in D^i$.

   (a) Define the *positive hull* of D^i, $\text{pos}(D^i)$, $i = 1, 2$. (Note this is sometimes called the conical hull or the cone generated by D^i).

   (b) Define the *convex hull* of P^i, $\text{conv}(P^i)$, $i = 1, 2$.

   (c) Prove that

   $$\cap_{i=1}^{2}(\text{pos}(D^i) + \text{conv}(P^i)) = \emptyset$$

   if and only if there exists a w such that

   $$\max_{d \in D^1} d^T w \leq \min_{d \in D^2} d^T w$$

   and

   $$\max_{p \in P^1} p^T w < \min_{p \in P^2} p^T w$$

   Be sure to quote any results you use precisely, and point out what happens if P^i is empty.

   (d) Explain how this is related to the strong separation of two polyhedral sets $\{x | A^i x \leq b^i\}$, $i = 1, 2$ in $\mathbf{R}^n$?

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2. We consider a portfolio allocation problem with a universe $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ of assets to purchase. For each stock $i \in N$, we are given its expected return α_i, and for each pair of assets $(i, j) \in N \times N$, the covariance between the (random) returns of these assets is given as the (symmetric, positive semi-definite) covariance matrix $Q \in \mathbb{R}^{n \times n}$. It can be shown that if $x \in \mathbb{R}^n$ is a vector of asset allocations, then $x^T Q x$ is the variance of the (random) return. Of course, $\sum_{i \in N} \alpha_i x_i$ is the expected return of the portfolio.

   (a) Write an optimization problem that maximizes the expected rate of return, subject to the "risk", as measured by the variance of the return of the portfolio, being no more than K. We must be fully invested, we have a budget of B, and there will be "no shorting," so that we must invest a non-negative quantity in each asset.

   (b) Now we are more interested in "portfolio rebalancing," where we have an initial portfolio $h \in \mathbb{R}^n$, with $e^T h = B$. ($e \in \mathbb{R}^n$ is the all-ones vector). We would like to add (linear) *transacation costs* to the previous model. Specifically, if we trade $t_i = |x_i - h_i|$ of asset i, then we pay an amount $c_i t_i$. Modify your model from (a) to maximize the expected rate of return minus the total transacation costs. Be sure to write the model so that the resulting model is (still) a quadratically-constrained program—(`qcp` in GAMS).

   (c) Now we would like to take a more active portfolio management strategy to help mitigate risk. There is a "sector" of stocks $S_1 \subset N$ such that if we decide to take a significant position in stocks in S_1 then we *also* want to take a significant position in a set $S_2 \subset N$ of less-risky stocks. Specifically, model the (logical) constraint that if more than 10% of the portfolio is held in stocks in S_1, then at least 5% of the portfolio must be allocated to stocks in S_2.

(d) The GAMS solvers for quadratically constrained programming are way too slow, so your client has demanded that you use *second-order-cone programming* (SOCP). Recall that a quadratic cone is the set of points:

$$C_q \stackrel{\text{def}}{=} \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R} \mid z \geq \|x\|_2 = \sqrt{\sum_{j=1}^n x_j^2}\right\},$$

and a *rotated* second order cone is the set of points

$$C_r \stackrel{\text{def}}{=} \{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^2_+ \; |2yz \geq \|x\|_2^2 = x^T x\}.$$

Change your model from problem (a) so that you no longer have a quadratic constraint, but rather your variable values are elements of an appropriate second-order-cone. *Hint:* You will need to add additional variables and use the fact that Q is positive semi-definite.

(e) (This one is harder). In order to reduce the "market impact" of large trades, instead of the linear transaction costs of problem (b), we would now like to assess a cost proportional to $t_i^{3/2}$. Specifically, if we trade $t_i = |x_i - h_i|$ of asset i, then we pay a "market impact" cost of $c_i t_i^{3/2}$. Modify your model from (b) to maximize the expected rate of return minus the total market-impact cost. This sounds easy, but it is actually difficult, because in order to get full credit, you must model the problem as a SOCP. *Hint:* You will need to add some auxiliary variables.

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3. Let $G = (V, A)$ be a directed graph with $n = |V|$ nodes and $m = |A|$ arcs. Let s and t be distinct nodes in V, and assume that an s-t path exists in G. Let $\chi^P, \; P \in \mathcal{P}$ be the set of incidence vectors of all s-t paths in G. (That is, $\chi^P_a = 1$ if arc $a \in A$ is on path P, and $\chi^P_a = 0$ otherwise.) The *dominant* of the s-t path set is:

$$X^{\uparrow}_{s-t} = \{x \in \mathbb{R}^m_+ \mid x \geq \chi^P \text{ for some } P \in \mathcal{P}\}.$$

Recall that a set of arcs $C \subseteq A$ is an s-t cut if $C = \{a = (i,j) \in A \mid i \in S, j \in V \setminus S\}$ for some $S \subset V$ with $s \in S$ and $t \notin S$.

(a) Let C be an s-t cut in G. Show that the inequality

$$\sum_{a \in C} x_a \geq 1 \tag{1}$$

is valid for $X^{\uparrow}_{s-t}$.

(b) Show that inequality (1) is *not* facet-defining for $\operatorname{conv}(X^{\uparrow}_{s-t})$ if C is a not a minimal s-t cut. (An s-t cut C is minimal if no strict subset of C is an s-t cut.) You may take it as given that $\operatorname{conv}(X^{\uparrow}_{s-t})$ is full-dimensional.

(c) Show that $\operatorname{conv}(X^{\uparrow}_{s-t})$ is equal to the set of $x \in \mathbb{R}^m_+$ that satisfy inequality (1) for all s-t cuts C in G. Hint: To show this, it is sufficient to show that whenever the following linear program has an optimal solution, it has an integral optimal solution:

$$\min \sum_{a \in A} c_a x_a \tag{2}$$

$$\text{subject to: } \sum_{a \in C} x_a \geq 1, \quad \forall s-t \text{ cuts } C \tag{3}$$

$$x \in \mathbb{R}^m_+. \tag{4}$$

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4. Show that the following three convex problems are equivalent. Carefully explain how the solution of each problem is obtained from the solution of the other problems. The problem data are the matrix $A \in \mathbb{R}^{m \times n}$ (with rows a_i^T), the vector $b \in \mathbb{R}^m$, and the constant $\mu > 0$.

(a) *Robust least-squares.*

$$\min \sum_{i=1}^{m} \phi(a_i^T x - b_i)$$

with variable $x \in \mathbb{R}^n$, where $\phi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\phi(u) = \begin{cases} u^2 & |u| \leq \mu \\ \mu(2|u| - \mu) & |u| > \mu \end{cases}.$$

(b) ℓ_1 *regularized least-squares.*

$$\min \|Ax - b - z\|_2^2 + 2\mu\|z\|_1$$

with variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$.

(c) *Weighted least-squares.*

$$\begin{array}{ll} \min & \sum_{i=1}^{m} (a_i^T x - b_i)^2/(w_i + 1) + \mu^2 \sum_{i=1}^{m} w_i \\ \text{s.t.} & w_i \geq 0 \qquad i = 1, \ldots, m \end{array}$$

with variables $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

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5. Prove the following separation results. You may use the fact that when X is a closed convex set, with $P(\cdot)$ denoting the operation of projection onto X, we have

$$(y - P(y))^T (z - P(y)) \leq 0 \text{ for all } y \text{ and for all } z \in X.$$

(a) Let X be nonempty, convex, and closed, with $0 \notin X$. Then there is $\bar{t} \in \mathbb{R}^n$ and $\alpha > 0$ such that $\bar{t}^T x \leq -\alpha$ for all $x \in X$.

(b) Let X and Y be two disjoint closed convex nonempty sets with X compact. Then there is $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $c^T x - \alpha < 0$ for all $x \in X$ and $c^T y - \alpha > 0$ for all $y \in Y$.

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6. In this problem, we consider the 2-stage stochastic program with *fixed* recourse

$$\min_{x \in \mathbb{R}^n} c^T x + \mathbb{E}_\xi[Q(x, \xi)] \qquad (5)$$
$$Ax = b, x \geq 0,$$

where, as usual the recourse function $Q(x, \xi) : \mathbb{R}^n \times \Xi \to \mathbb{R} \cup \{+\infty\}$ is given as

$$Q(x, \xi) = \min_{y \in \mathbb{R}^m} q^T y$$
$$Tx + Wy = h, y \geq 0$$

where $\xi = (q, h, T)$ is a random vector, with components $q \in \mathbb{R}^n_+, h \in \mathbb{R}^p$, and $T \in \mathbb{R}^{p \times n}$. Note that the matrix W is not random.

(a) Give the definition of the problem (5) having *complete recourse*

(b) Suppose that the (fixed) matrix W has the following property:

$$\mathbf{P} : \quad \{\pi \in \mathbb{R}^q \mid W\pi \leq 0\} = \{0\}.$$

True of False. If property **P** holds, then the problem (5) has *complete recourse.* If true, provide a proof. If false, provide a counterexample.
