MATHEMATICAL PROGRAMMING

Instructions for Depth Exam Students: Answer any 6 of the 8 following questions.

Instructions for Breadth Exam Students: Answer any 3 of the 8 following questions.

1. Use the simplex method to solve the following problem:

$$\begin{array}{llrcrcrcr} \text{maximize} & & & & 5x_1 - x_2 + 11x_3 & & & & \\ \text{subject to} & & x_1 & - & x_2 & + & 2x_3 & \leq & 3 \\ & & -2x_1 & + & x_2 & - & 2x_3 & \leq & 8 \\ & & \multicolumn{7}{c}{-2 \leq x_1 \leq 3, \quad -1 \leq x_2 \leq 1, \quad x_3 \leq 0} \end{array}$$

2. Consider

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_i x \geq b_i \quad i = 1, \ldots, m \end{array} \tag{1}$$

Let x^1 be optimal for

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_i x \geq b_i \quad i = 2, \ldots, m \end{array}$$

and x^2 be optimal for

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_1 x = b_1 \\ & A_i x \geq b_i \quad i = 2, \ldots, m \end{array}$$

(a) Prove that (**??**) is solvable.

(b) Give an upper and a lower bound for the minimum value of (**??**).

(c) Prove that one of x^1 or x^2 solves (**??**).

3. Let

$$\theta(b) := \min\{f(x) \mid g(x) \leq b\} \tag{2}$$

where $f\colon \mathrm{I\!R}^{\mathrm{n}} \mapsto \mathrm{I\!R}$, $g\colon \mathrm{I\!R}^{\mathrm{n}} \mapsto \mathrm{I\!R}^{\mathrm{m}}$ are convex functions on $\mathrm{I\!R}^{\mathrm{n}}$ and $b \in \mathrm{I\!R}^{\mathrm{m}}$. Suppose that for each b in some set B in $\mathrm{I\!R}^{\mathrm{m}}$, the minimization problem of (**??**) is solvable. Suppose now that an optimal Lagrange multiplier $u(\bar{b}) \in \mathrm{I\!R}^{\mathrm{m}}_{+}$ exists for some fixed $\bar{b} \in B$, such that in addition to satisfying the Kuhn-Tucker saddlepoint conditions, $u(\bar{b})$ also satisfies

$$u(\bar{b})(\bar{b} - b) \geq 0 \quad \forall b \in B$$

What can you say about $\theta(\bar{b})$ relative to $\theta(b)$ for all $b \in B$? Prove your claim.

4. Suppose that for some real number $\gamma > 0$, $x(\gamma) > 0$ solves the interior penalty problem

$$\min \left\{ f(x) - \gamma \sum_{j=1}^{n} \log x_j \;\middle|\; Ax = b \right\}$$

where $f\colon \mathrm{I\!R}^{\mathrm{n}} \mapsto \mathrm{I\!R}$, A is an mxn real matrix, b is an mx1 real vector and f is convex and differentiable on $\mathrm{I\!R}^{\mathrm{n}}$. Give a lower bound to

$$\inf\{f(x) \mid Ax = b,\ \ x \geq 0\}$$

in terms of $f(x(\gamma))$, γ and n. Establish your claim.

5. Suppose the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0 \end{array} \tag{3}$$

(where $f\colon \mathrm{I\!R^n} \mapsto \mathrm{I\!R}$ and $h\colon \mathrm{I\!R^n} \mapsto \mathrm{I\!R^m}$ are continuous functions) has a solution x^*. Let M be an optimistic estimate of $f(x^*)$, that is, $M \leq f(x^*)$. Consider the unconstrained problem

$$\underset{x}{\text{minimize}} \quad v(M,x) := [f(x) - M]^2 + \|h(x)\|^2 \tag{4}$$

Consider the following algorithm. Given $M_k \leq f(x^*)$, a solution x_k to problem (**??**) with $M = M_k$ is found, then M_k is updated by

$$M_{k+1} = M_k + [v(M_k, x_k)]^{1/2} \tag{5}$$

and the process repeated.

(a) Show that if $M = f(x^*)$, a solution of (**??**) is a solution of (**??**).

(b) Show that if x_M is a solution of (**??**), then $f(x_M) \leq f(x^*)$.

(c) Show that if $M_k \leq f(x^*)$ then M_{k+1} determined by (**??**) satisfies $M_{k+1} \leq f(x^*)$.

(d) Show that $M_k \longrightarrow f(x^*)$.

6. Let f be a semipositive vector in $\mathrm{I\!R^n}$ (that is, each component of f is non-negative, and $f \neq 0$), and define a function ϕ from $\mathrm{I\!R^n}$ to the extended reals by

$$\phi(x) = \inf\{\, \alpha \mid x \leq \alpha f \,\}.$$

(a) Show that ϕ is a closed proper convex function.

(b) Compute the conjugate function ϕ^* and show that ϕ^* is the indicator of a certain polyhedral convex set S. Exhibit S explicitly.

(c) Explain what the support function of a set is. Prove that ϕ is a support function, and identify the set of which it is the support function.

(d) Give a condition on f (only) to ensure that the effective domain of ϕ^* is compact. Justify your condition.

7. Let

$$\begin{array}{ll} \underset{x}{\text{minimize}} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

be a min cost network flow problem

(a) State the corresponding "big M" problem, being precise about how M is obtained from the data of the original problem.

(b) Show that if the **big M problem is unbounded**, then the original problem is either unbounded or infeasible.

(c) What **additional** procedure may be performed to identify which of the alternatives in (b) holds?

8. Write an integer program equivalent to

$$\begin{array}{ll} \text{maximize} & z = 3x_1 + 4x_2 - 3x_3 \\ \text{subject to} & x_1 + x_2 + 4x_3 \leq 60 \\ & -x_1 + 2x_2 + x_3 \geq 12 \\ & \text{if } x_2 + x_3 > 0 \text{ then } x_1 + x_3 \leq 54 \\ & (x_1, x_2, x_3) \geq 0 \end{array}$$