**MATHEMATICAL PROGRAMMING**

**Depth Exam: Answer any 6 of the following 8 questions**

**Breadth Exam: Answer any 3 of the following 8 questions**

1. Solve by the simplex method the following Linear Programming Problem:

$$
\begin{array}{llllllllll}
\max & & - 4x_2 - 2x_3 - 7x_4 & & & & & & & \\
 & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & = & 3 \\
\text{s.t.} & x_1 & - & x_2 & & & + & x_4 & \leq & 1 \\
 & 2x_1 & + & 2x_2 & & & + & 4x_4 & \leq & 3 \\
 & & & x_i \geq 0 & & i = 1, \ldots, 4 & & & &
\end{array}
$$

Hint: Start with $x_3$ as a basic variable even though it also occurs in the objective function.

3. Let $f : R^n \to R$ be a differentiable convex function on $R^n$ and let $x^1, \ldots, x^k$ be $k$ points in $R^n$ such that for some $\bar{u} \in R^k$ :

$$\sum_{i=1}^{k} \bar{u}_i \nabla f(x^i) = 0, \ \sum_{i=1}^{k} \bar{u}_i = 1, \ \bar{u} \geq 0$$

Give a lower bound to $\inf_{x \in R^n} f(x)$ in terms of $x^1, \ldots, x^k$ and $\bar{u}$.

6. Suppose that $G$ is a directed graph consisting of $n$ nodes and $K$ (connected) components. Prove that the rank of the corresponding node-arc incidence matrix is $n - K$.

7. Consider the integer program

$$(IP) \qquad \begin{array}{ll} \min\limits_{x} & cx \\ \text{s.t.} & Ax = b \\ & x_i = 0 \text{ or } 1 \quad (i = 1, \ldots, n) \end{array}$$

Assuming (IP) is feasible, prove that for sufficiently large $M$, every optimal solution of (IP) is also an optimal solution of

$$(NLP) \qquad \begin{array}{ll} \min\limits_{x} & cx + M \sum x_i(1 - x_i) \\ \text{s.t.} & Ax = b \\ & 0 \le x_i \le 1 \quad (i = 1, \ldots, n). \end{array}$$

8. Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ a & \text{if} \quad x \in (0, 1] \\ bx + c & \text{if} \quad x \in (1, 2] \end{cases}$$

where $a, b$ and $c$ are real numbers. Determine conditions on $a, b$ and $c$ such that

(a) there exists a Linear Program Minimization Model for $f(x)$ on $[0, 2]$;

(b) there exists a Mixed Integer Minimization Model for $f(x)$ on $[0, 2]$.

Write a Mixed Integer Minimization Model for $f(x)$ on $[0, 2]$ (under conditions found in (b)). ∎

**Convex programming question Spring 90**

Suppose $f$ is a closed proper convex function on $\mathbf{R}^n$, and let $K$ be a nonempty closed convex cone, also in $\mathbf{R}^n$. Consider the primal problem of minimizing $f$ on $K$. Using the duality structure given by

$$F(x, p) = \begin{cases} f(x) & \text{if } x \in K + p, \\ +\infty & \text{otherwise,} \end{cases}$$

obtain the Lagrangian and the dual objective function for this problem in the simplest practical form. Also prove that the dual problem has a maximum (attained) if

$$\operatorname{ri} K \cap \operatorname{ri} \operatorname{dom} f \neq \emptyset.$$