MATHEMATICAL PROGRAMMING

Depth Exam: Answer **6** questions, with at most **2** questions from **1, 2, 3.**
Breadth Exam: Answer **3** questions, with at most **2** questions from **1, 2, 3.**

1. Consider the following LP:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b, \;\; x \geq 0. \end{aligned}$$

   (a) Assuming that the feasible set is <u>bounded</u>, prove that if the optimal solution is <u>not unique</u>, then there exist <u>at least two</u> vertices of the feasible set that are optimal.

   (b) If the boundedness hypothesis is deleted in part (a), is the result still valid? Prove or provide a numerical counterexample.

2. Use the simplex method to solve the linear programming problem with lower and upper bounds:

$$
\begin{array}{lllllllll}
\max & z = & x_1 & & & + & 3x_3 & & \\
\text{s.t.} & & x_1 & + & x_2 & - & x_3 & & = 1 \\
& & 2x_1 & & & - & 4x_3 & + \ x_4 & = 1 \\
& & \multicolumn{7}{l}{0 \le x_1 \le 5,\ 0 \le x_2 \le 2,} \\
& & \multicolumn{7}{l}{0 \le x_3 \le 3,\ x_4 \ge 0}
\end{array}
$$

3. Show that $\bar{x} = (1, 0, 1)$ is an optimal solution of the quadratic program

$$
\begin{array}{ll}
\max & \frac{1}{2}x^T Q x + c^T x \\
\text{s.t.} & Ax \le b \\
& x \ge 0,
\end{array}
$$

where

$$
Q = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}
$$

$$
b = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{c}^{\mathrm{T}} = (2,\ 2,\ 2).
$$

Be careful to quote any results that you use accurately.

4. Let (x^1, u^1) and (x^2, u^2) be distinct points in $\mathbf{R}^{n+m}$ which satisfy the Karush-Kuhn-Tucker **saddlepoint** conditions for the problem

$$
\min \ f(x) \quad \text{s.t. } g(x) \le 0
$$

where $f\colon \mathbf{R}^n \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}^m$.

(a) What can you say about (x^1, u^2) and (x^2, u^1)? Establish your claim.

(b) If g is linear and f is differentiable, how are $\nabla f(x^1)$ and $\nabla f(x^2)$ related? Prove your claim.

5. Let $F\colon S \to S$ where S is a closed set in R^n and let

$$
\|F(y) - F(x)\| \le \gamma \|y - x\|, \ \forall x,\ y \in S \ \text{ and some } \gamma < 1.
$$

(a) Consider an algorithm $x^{i+1} = F(x^i), \ \forall i \geq 0, \ x^0 \in S$. Prove that $\exists x^* \in S$ such that $x^i \to x^*$ and $x^* = F(x^*)$.

(b) Consider the case where

$$F(x) = \frac{2x^3 + a}{3x^2}, \quad a > 0.$$

Prove that if $x > 0$, then $F(x) \geq \sqrt[3]{a}$. Using this fact to determine an appropriate S, show that the method converges from any starting point $x^0 > 0$ to the cube root of a.

(c) Explain briefly why the method is locally quadratically convergent.

6. Consider the problem

$$\min \ f(x) \quad \text{s.t.} \ g(x) \leq 0$$

where $f: \mathbf{R}^n \to \mathbf{R}$ and $g: \mathbf{R}^n \to \mathbf{R}^m$.

(a) Write one complete step of the Augmented Lagrangian Algorithm.

(b) Give a lower bound to $\inf \{f(x) \big| g(x) \leq 0\}$ in terms of the current iterate of the Augmented Lagrangian Algorithm.

(c) What can you say about a fixed point of the Augmented Lagrangian Algorithm?

7. Suppose that f and $g_1, \ldots, g_m$ are closed proper convex functions from $\mathbf{R}^n$ to $\mathbf{R}$, all having the same effective domain D. Consider the parametrized optimization problem

$$\inf f(x) + \langle u, x \rangle, \text{ subject to } g_i(x) \leq v_i \ (i = 1, \ldots, m), \qquad (\mathrm{P}(u, v))$$

where u and v are prescribed vectors in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively.

For any particular choice of functions f and $g_1, \ldots, g_m$ as above, do there always exist values $\hat{u}$ and $\hat{v}$ of the parameters such that *both* the problem $\mathrm{P}(\hat{u}, \hat{v})$ and its dual with respect to the standard right-hand side parametrization have optimal solutions? Justify your answer.

8. Two machines M_1 and M_2 are available for at most 8 and 10 hours respectively for the production of two products A and B, that may be sold for \$10 and \$11 per unit respectively. The production rates (units per hour) are given in the following table:

Sales are limited to at most 20 units of A and 50 units of B.

(a) Formulate the problem of maximizing total revenue as a generalized network flow problem. Give both an algebraic and a graphical formulation.

(b) If a parallel computer is available, describe how, starting with an all-slack basis, two pivots may be performed in parallel, producing an improved basic feasible solution. Show the flows for this basic feasible solution graphically.