MATHEMATICAL PROGRAMMING
Spring 1994 Qualifying Exam

Depth Exam: Answer 6 questions, with at most 2 questions from 1,2,3.
Breadth Exam: Answer 3 questions, with at most 2 questions from 1,2,3.

1. Find an upper bound, in terms of the problem data, to the maximum value of the following feasible linear program:

$$\text{maximize } p^T x \text{ subject to } Ax \leq b,\ e^T x \leq 1,\ x \geq 0$$

Here $p \in R^n$, $A \in R^{m \times n}$, $b \in R^m$, and e is a vector of ones in R^n.

2. Consider the following bounded-variable LP with a single general linear constraint:

$$\begin{array}{ll} \underset{x_i}{maximize} & \displaystyle\sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} a_i x_i = b \\ & 0 \leq x_i \leq u_i \quad i = 1, \ldots n \end{array}$$

(a) State an LP dual in which there is a dual variable corresponding to the equation and a dual variable for each upper bound constraint.

(b) Assume that $a_i > 0$ and $u_i > 0 \quad i = 1, \ldots n$ and that the variables have been indexed so that $c_i/a_i \geq c_{i+1}/a_{i+1} \quad i = 1, \ldots n-1$. If $\sum_{i=1}^{n-1} a_i u_i \quad + a_n u_n/2 = b$, state primal and dual optimal solutions and verify that objective values of these two solutions are identical. (Hint: The preceding expression for b determines a primal optimal solution.)

3. Compute the optimal solutions of the following three problems in terms of c:

$$\min\{c^T x : \|x\|_1 \leq 1\},$$

$$\min\{c^T x : \|x\|_2 \leq 1\},$$

$$\min\{c^T x : \|x\|_\infty \leq 1\}.$$

In addition, reformulate just the first problem as an LP, construct its dual and write down the solution of the dual.

4. Let $X = \{x \mid g(x) \leq 0,\ x \geq 0\} \neq \emptyset$, where $g : R^n \to R^m$ is a differentiable convex function on R^n. Suppose that there exist $\bar{x} \in R^n$ and $\bar{u} \in R^m$ such that: $\bar{u} \geq 0,\ \bar{u}\nabla g(\bar{x}) > 0$. Does the problem $\max_{x \in X} \|x\|_1$, have a solution? Justify your answer.

5. Suppose that f is continuously differentiable and there exists a constant K such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \le K\|x - y\|_2$$

for all x, $y \in C$, a closed convex subset of R^n. Let

$$x(\alpha) = \pi_C(x - \alpha\nabla f(x))$$

where π_C denotes the projection operator onto C, $\alpha > 0$ and $x \in C$. Show that

(a) $f(x(\alpha)) \le f(x) - (1/\alpha - K/2)\|x(\alpha) - x\|_2^2$.

(b) $x(\alpha) = x$ if and only if $(y - x)^T\nabla f(x) \ge 0$, for all $y \in C$.

(c) If f is also convex on C, then $x(\alpha) = x$ if and only if x solves $\min_{x\in C} f(x)$.

6. Let f be a closed proper convex function on R^n. Suppose that there is an open, bounded convex set Q in R^n and a point $x_0 \in Q$ such that for each point x on the boundary of Q one has $f(x) > f(x_0)$.

Now let $v \in R^n$ be arbitrary and let ϕ be any real number greater than $\inf f$. Consider the optimization problem of minimizing the inner product $\langle v, x\rangle$ over all x satisfying the single constraint $f(x) \le \phi$. Prove that this problem has a finite optimal value and that this value is attained at some feasible point x.

7. Consider the following problem defined with respect to the nodes and arcs of a digraph (N, A) where x_{ij} is the flow on arc (i,j):

$$\begin{array}{ll} \underset{x}{minimize} & cx \\ \text{subject to} & \displaystyle\sum_{j\ni\ (i,j)\in A} x_{ij} - \sum_{j\ni\ (j,i)\in A} x_{ji} \le b_i \quad for \quad i \in N_1 \\ & 0 \le x_{ij} \le u_{ij} \quad for \quad (i,j) \in A \end{array}$$

where N_1 is a nonempty proper subset of N.

State an equivalent problem in standard network flow form, i.e., construct an appropriate digraph (N', A') and appropriate data such that the preceding problem is equivalent to

$$\begin{array}{ll} \underset{y}{minimize} & c'y \\ \text{subject to} & \displaystyle\sum_{j\ni\ (i,j)\in A'} y_{ij} - \sum_{j\ni\ (j,i)\in A'} y_{ji} = d_i \quad for \quad i \in N' \\ & 0 \le y_{ij} \le u'_{ij} \quad for \quad (i,j) \in A' \end{array}$$

Note that all nodes in N' appear in the divergence equations. Be sure to discuss what may be concluded about variables in the original problem that do not have corresponding variables in the equivalent problem.

8. Consider the following IP:

$$\begin{array}{rl} \mathit{minimize} & cx \\ & Ax = a \\ \text{subject to} & Dx = d \\ & x \in \mathbb{Z}^n_+ \end{array}$$

Let

$$L_1(\lambda) := \min_x \begin{array}{l} cx - \lambda(Ax - a) \\ Dx = d \\ x \in \mathbb{Z}^n_+ \end{array} \qquad L_2(\mu) := \min_{x,y} \begin{array}{l} \frac{1}{2}cx + \frac{1}{2}cy - \mu(x - y) \\ Ax = a \\ Dy = d \\ x \in \mathbb{Z}^n_+ \\ y \in \mathbb{Z}^n_+ \end{array}$$

Show that

$$z^* \leq L_1^* := \max_\lambda L_1(\lambda) \leq L_2^* := \max_\mu L_2(\mu) \leq z^{**}$$

where z^{**} is the optimal value of the IP and z^* is the optimal value of the linear program relaxation of the the IP.

Hint: You may want to use that fact that for the problem min $\{rx,\ Rx = r,\ x \in X\}$, where $X = \left\{x \in \mathbb{Z}^n_+,\ Qx = q\right\}$, the optimal objective function value of the Lagrangian relaxation problem (when the constraint $Rx = r$ is dualized) equals the optimal objective function value of the linear program min $\{rx, Rx = r, x \in conv(X)\}$. Note also that $L_2(\mu)$ represents relaxation of the constraint $x = y$.