# MATHEMATICAL PROGRAMMING

Spring 1997 Qualifying Examination

**Answer 5 questions from the following 7 questions.**

1. Solve the following quadratic program:

$$\begin{array}{ll} \min & \frac{1}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 + 8x_1 - x_2 + 5 \\ \text{subject to} & x_1 - 2x_2 \geq 2 \\ & x_1 - x_2 \geq -7 \\ & x_1, x_2 \geq 0 \end{array}$$

2. Consider the linear programming problem

$$\begin{array}{llllllll} \text{minimize} & -3x_1 & +(4-\theta)x_2 & -x_3 & +15x_4 & & \\ \text{subject to} & x_1 & +2x_2 & +x_3 & +2x_4 & = & 2 \\ & -2x_1 & & -x_3 & +5x_4 & = & -3 \\ & x_j \geq 0 \text{ for all } j. & & & & & \end{array}$$

Determine the optimal $x$ and the optimal objective value for each real value of $\theta$. Show your work. (Suggestion: Start with $\theta = 0$ and the basis $\{x_1, x_3\}$.)

3. Let $L$ be a nonempty subspace of $\mathbb{R}^n$ and let $f$ be a closed proper convex function on $\mathbb{R}^n$. Show that if $L \cap \operatorname{ri} \operatorname{dom} f \neq \emptyset$, then

$$\inf_{x \in L} f(x) = \max_{x^* \in L^\perp} -f^*(x^*),$$

where the maximum on the right indicates attainment (but not necessarily at a finite value), and where $f^*$ denotes the convex conjugate of $f$. Give a specific example in which the maximum is attained at an infinite value.

In answering this question you may use without proof standard theorems and constructions of convex analysis, but if you do so you must identify the theorem or construction you are using.

4. Let $f : \mathbf{R}^n \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable convex functions on $\mathbf{R}^n$ and let $(\bar{x}, \bar{u}) \in \mathbf{R}^{n+m}$ be a Karush-Kuhn-Tucker point of

$$\min f(x) \text{ subject to } g(x) \leq 0$$

Let $g(x^1) < 0$ and let $(x^2, u^2)$ be dual feasible, that is

$$\nabla f(x^2) + u^2 \nabla g(x^2) = 0, \quad u^2 \geq 0$$

Give an upper bound on $\|\bar{u}\|_1$ in terms of $(x^1, x^2, u^2)$.

5. Consider the problem

$$\min_{x \in X} := \min_{x} \{f(x) : Ax \geq b, \quad x \geq 0\}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable *concave* function on $\mathbf{R}^n$ which is bounded below on the nonempty set $X$. Consider the successive linearization algorithm

$$x^{i+1} \in \arg \text{vertex} \min_{x \in X} \nabla f(x^i)(x - x^i); \quad \text{stop if } \nabla f(x^i)(x^{i+1} - x^i) = 0$$

Show that this algorithm is well defined and that $\{x^i\}$ terminates at some $x^{\bar{i}}$ which satisfies the minimum principle necessary optimality condition:

$$\nabla f(x^{\bar{i}})(x - x^{\bar{i}}) \geq 0 \quad \text{for all } x \in X.$$

6. Let

$$A := \{x \mid x_1 + 2x_2 + 3x_3 \leq 6, x \geq 0\}$$

and

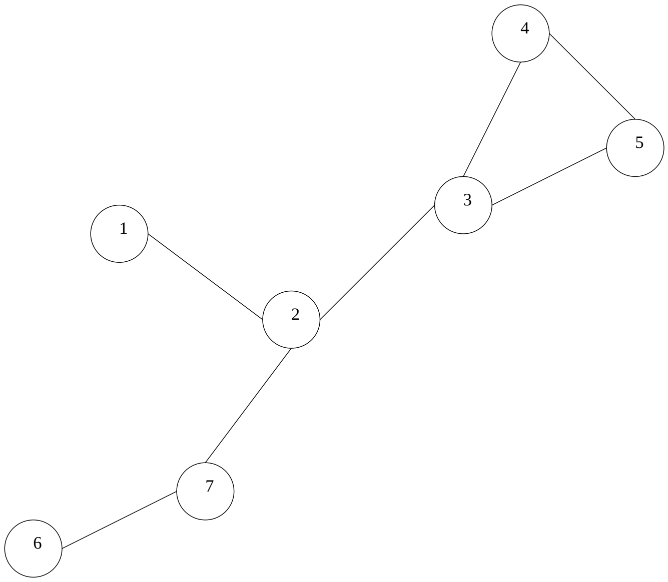
$$B := \{x \mid 3x_1 + 2x_2 + x_3 \leq 12, x \geq 0\}.$$

Describe how constants r and s may be determined such that $A \cup B$ is the projection onto the x-space of the set

$$\{(x, y) \mid x_1 + 2x_2 + 3x_3 \leq 6 + ry, 3x_1 + 2x_2 + x_3 \leq 12 + s(1 - y),$$
$$x \geq 0, \ y \text{ binary.}\}.$$

Determine the *smallest* numerical values of r and s such that this projection relationship holds (show your work). Discuss why the projection relationship does not hold if either value is reduced.

7. The following network represents an electrical power distribution network connecting power generating points with power consuming points.




The arcs are undirected; that is, power may flow in either direction. Points 1,4 and 7 are generating points with generating capacities and unit costs given by the following

| | 1 | 4 | 7 |
|---|---|---|---|
| Capacity (in thousands of kilowatt hours) | 100 | 60 | 80 |
| Unit cost ($ per thousand KWH) | 15.0 | 13.5 | 21.0 |

Points 2, 5 and 6 are consuming points with demands of 35,000, 50,000, and 60,000 KWH respectively. There is no upper bound on transmission line capacity and the unit cost of transmission on each line segment is $11.00 per 1000 KWH.

(a) Set up this problem as a minimum cost network flow problem, using a carefully labeled diagram. Make sure you have directed arcs in your formulation and the sum of the divergences is zero.

(b) Write down a solution to the resulting problem and verify its optimality.